Rings & Fields I — Spring 2011

Exam B

Instructions:
Please work out each problem starting on a new sheet of paper. Mark each sheet carefully with your name, the problem it relates to, and which number sheet it is.

1. Prove or disprove the following statments
   (3pts) (i) A homomorphic image of a field is a field.
   (3pts) (ii) A subring of a field is a field.
   (4pts) (iii) The Cartesian product of two fields is a field.

2. Let $F = \mathbb{Z}_3$.
   (a) (2pts) Prove that $f(x) = x^2 + 2x + 1$ is irreducible in $F[x]$.
   (b) (8pts) Find a representative for each of the 9 elements of $F[x]/<f(x)>$ and write down the Cayley table for both the addition and the multiplication. Is this ring a field?

3. (1pts) (a) Give the definition of a Principal Ideal Domain (PID);
   (5pts) (b) Prove that if $h : R \to S$ is a surjective ring homomorphism and R is a PID then so is S;
   (4pts) (c) Prove that $\mathbb{Z}_m$ is a PID for each natural number m.

4. Determine whether or not homomorphisms with each of the following properties exist. If they do, define them.
   (5pts) (a) A homomorphism from $\mathbb{Q}[x]$ to $\mathbb{C}$ with kernel generated by $x^2 - x - 6$;
   (5pts) (b) A homomorphism from $\mathbb{Q}[x]$ to $\mathbb{C}$ with kernel generated by the two polynomials $x^4 + 2x^3 - x^2 + 2x - 2$ and $x^3 + 2x^2 + x + 2$;

5. Let $C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ be the ring of real valued continuous functions on the unit interval with pointwise addition and multiplication.
   (5pts) (a) Is $C[0,1]$ an integral domain?
   (5pts) (b) Find a maximal ideal, M, of $C[0,1]$ and determine the quotient $C[0,1]/M$.

6. Let $R = \{a + bi \mid a, b \in \mathbb{Z}\}$.
   (5pts) (a) Is $M = \{a + bi \mid 5|a \text{ and } 5|b\}$ a maximal ideal of R?
   (5pts) (b) Is $M =< 2 + i >$ a maximal ideal of R? Describe R/M in terms of rings that you know.